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Two-Dimensional Space Groups with Sevenfold Symmetry

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Abstract

Fivefold and sevenfold symmetry operations are, of course, incompatible with repetition by a lattice but, with the appearance of structures involving curved sheets, they and other non-crystallographic operations must now be taken into consideration as possibilities of non-Euclidean crystallography develop. Here are described the symmetry groups which might be called 732 and 73m and which may be found in two-dimensional manifolds.

Structures with fivefold symmetry on the surface of a sphere have been familiar at least since the work of Caspar & Klug (1962). There are two ways of regarding the packing of units on the surface of a sphere:

(a) they may be considered as forming a finite particle with the three-dimensional point symmetry groups 532 or 53m for which there are respectively (depending on whether mirror symmetry is forbidden or permitted) 60 and 120 fundamental regions or asymmetric units of pattern; or

(b) they may be considered as a packing in two-dimensional curved space of non-Euclidean metric. This approach enables us to bring sevenfold (and higher) symmetry within the compass of crystallography.

The curvature arises because five units, which may be equilateral triangles, pack around a fivefold axis to give an icosahedron. If the edges of this icosahedron are projected radially on to the circumscribed sphere then a tessellation of spherical triangles is obtained. The sum of the angles of a triangle made up of geodesics on the surface is given by

$$\alpha+\beta+\gamma=\pi+\int K\,\mathrm{d}S,$$

where K is the Gaussian curvature of the surface. As the curvature is positive the surface closes on itself and is of finite area.

At any point in a surface there will be two principal curvatures K_1 and K_2 in perpendicular planes. The mean (or first) curvature is $J=(K_1+K_2)/2$ and the Gaussian (or second) curvature is $K=K_1K_2$.

In any extended plane tessellation of triangles, the mean coordination number of a point is six. If the coordination number is less than six then a spherical or positively curved elliptical space is obtained. However, if the mean coordination number is greater than six a curved two-dimensional space is obtained having hyperbolic or negative Gaussian curvature. A graphic illustration of a surface where the local coordination in the surface is greater than six is provided by a frond of crinkled seaweed such as *Fucus letuca*, where the area out to a distance r from any given point increases faster than πr^2. In fact, the circumference of a small circle on a surface of Gaussian curvature K is given by $s(r)=2\pi r-(1/3)\pi K r^3+$terms in r^5 and higher powers. If the space is curved and non-Euclidean then the parallel postulate of Euclid fails and the concept of repetition on a lattice must be abandoned. On the surface of a sphere the asymmetric units are repeated by rotations and reflections. In addition to the groups 532 and 53m, the axial groups N, $N2$, N/m are well known.

The surface of a sphere is, of course, a finite space but surfaces with negative Gaussian curvature may be infinite. Some of these have recently come into

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practical prominence with their appearance as building elements in lipid and silicate structures (Andersson, 1983; Longley & McIntosh, 1983; Mackay, 1979, 1985). Periodic minimal surfaces, first described by Schwarz (1890), are surfaces of infinite extent which have $J = 0$ so that the Gaussian curvature K is everywhere non-positive. Just as for the surface of a sphere, a planar structure, such as a lipid bilayer, can only be mapped on to such a surface if disclinations are regularly introduced (Mosseri & Sadoc, 1982; Gaspard, Mosseri & Sadoc, 1983; Nelson, 1983).

We wish to point out now that a two-dimensional surface with constant hyperbolic curvature can, for example, be tessellated with an infinite packing of identical regular heptagonal cells (Fig. 1) and that classical crystallography can be extended in this direction. There are two different groups, which might be named $73m$ and 732, respectively with and without mirror planes. The axes of symmetry are sufficient to repeat the unit of symmetry and, as on the surface of a sphere, translation operations are not necessary.

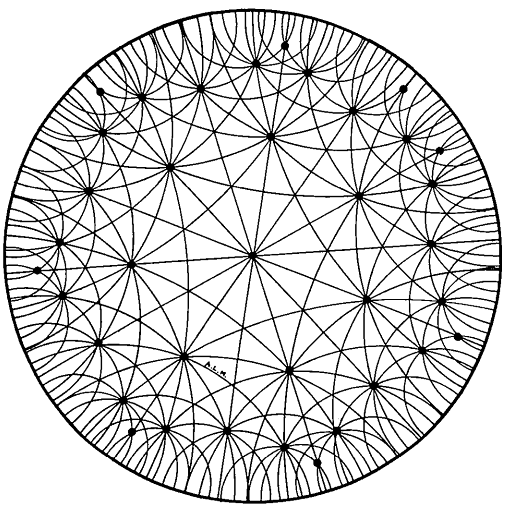

Fig. 1. The stereographic projection of a two-dimensional manifold of constant negative Gaussian curvature. The fundamental regions are triangles with angles $\pi/2$, $\pi/3$ and $\pi/7$ and 14 of these make up a regular heptagon, which is repeated to cover the infinite space according to the space group $73m$.

An infinite space of constant negative Gaussian curvature can be represented in stereographic projection (Hilbert & Cohn-Vossen, 1952). Angles are preserved in the projection. It can be seen that the fundamental region is a triangle with angles $\pi/2$, $\pi/7$ and $\pi/3$. These angles leave a spherical deficit per triangle of $\pi/42$, corresponding to what is the least-curved two-dimensional manifold.

Other similar groups, which might be named 642 and $6m42$, where the angles between geodesics of the fundamental triangle are $\pi/6$, $\pi/4$, and $\pi/2$, with an angular deficit of $\pi/12$, have been drawn by Coxeter (1961). The groups 542 and $5m42$ with angles in the fundamental triangle of $\pi/5$, $\pi/2$ and $\pi/4$, with an angular deficit of $\pi/20$ have been illustrated by Schwarz (1890) and there are infinitely more. Such hyperbolic tessellations (with q p-gons meeting at a point) can occur when $1/p + 1/q < 1/2$ (Somerville, 1914).

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